EXTRINSIC RADIUS PINCHING FOR HYPERSURFACES OF SPACE FORMS

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Abstract. We prove some pinching results for the extrinsic radius of compact hypersurfaces in space forms. In the hyperbolic space, we show that if the volume of M is 1, then there exists a constant C depending on the dimension of M and the L^∞-norm of the second fundamental form B such that the pinching condition $\tanh(R) < \frac{1}{||H||_\infty} + C$ (where H is the mean curvature) implies that M is diffeomorphic to an n-dimensional sphere. We prove the corresponding result for hypersurfaces of the Euclidean space and the sphere with the L^p-norm of H, $p \geq 2$, instead of the L^∞-norm.

Key words: Extrinsic radius, pinching, hypersurfaces, space forms
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1. Introduction

Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into the $(n+1)$-dimensional simply connected space-form $(\mathbb{M}^{n+1}(\delta), g_{can})$ of sectional curvature δ with $n \geq 2$. First, let us recall the definition of the extrinsic radius of M.

Definition 1.1. *The extrinsic radius of (M, g) is the number*

$$R = R(M) = \inf \left\{ r > 0 \middle| \, \exists x \in \mathbb{M}^{n+1}(\delta) \ s.t. \ \phi(M) \subset B(x, r) \right\},$$

where $B(x, r)$ is the open ball of center x and radius r in $\mathbb{M}^{n+1}(\delta)$.

Throughout this paper, we denote respectively by $B(x, r)$, $\overline{B}(x, r)$ and $S(x, r)$ the open ball, the closed ball and the sphere of center x and radius r in $\mathbb{M}^{n+1}(\delta)$. An immediate consequence of the above definition is that there exists $p_0 \in \mathbb{M}^{n+1}(\delta)$ such that $\phi(M) \subset \overline{B}(p_0, R)$ and $\phi(M) \cap S(p_0, R) \neq \emptyset$. Moreover, it is a well-known fact that the extrinsic radius is bounded from below in terms of the mean curvature. More precisely, we have the following estimate due to Hasanis and

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Koutroufiotis ([10]) for $\delta = 0$ and Baikoussis and Koufogiorgos ([2]) for any δ

$$t_\delta(R) \geq \frac{1}{||H||_\infty}, \tag{1}$$

where t_δ is the function defined in Section 2 and H the mean curvature of the immersion. Note that for $\delta > 0$, the image $\phi(M)$ is assumed to be contained in a ball of radius less than $\frac{\pi}{2\sqrt{\delta}}$, that is an open hemisphere. Moreover, equality in (1) is characterized by geodesic hyperspheres.

A natural question is the following: Is there a constant C, depending on a minimal number of geometric invariants, such that if we have the pinching condition

$$t_\delta(R) < \frac{1}{||H||_\infty} + C, \tag{P_C}$$

then M is closed, in a certain sense, to a sphere?

Many pinching results are known for geometric invariants defined on Riemannian manifolds with positive Ricci curvature, as the intrinsic diameter ([7, 15, 19]), the volume, the radius ([5, 4]) or the intrinsic lower bound of Lichnerowicz-Obata of the first nonzero eigenvalue of the Laplacian in terms of lower bounds of the Ricci curvature ([6, 15, 16]).

For instance, concerning the intrinsic diameter, under the hypothesis that (M^n, g) is a complete Riemannian manifold with $\mathrm{Ric} \geq n-1$, Myers gave the well-known upper bound

$$\mathrm{diam}(M^n, g) \leq \mathrm{diam}(\mathbb{S}^n, can) = \pi.$$

In particular, M is a compact manifold.

S. Ilias proved in [15] that there exists an ε depending on n and an upper bound of the sectional curvature so that if $\mathrm{Ric} \geq n-1$ and $\mathrm{diam}(M) > \pi - \varepsilon$, then M is homeomorphic to $\mathbb{S}^n$.

Petersen and Sprouse gave in [17] a generalization of the Theorem of Myers with a less restrictive assumption on the Ricci curvature. They assume that Ric is almost bounded from below by $n-1$ in an L^p-sense. Then under this hypothesis, $\mathrm{diam}(M^n, g) \leq \pi + \varepsilon$.

With a similar hypothesis on the Ricci curvature, E. Aubry ([1], Theorem 5.24) proved that if $\mathrm{diam}(M^n, g) \geq \pi - \varepsilon$ for ε small enough depending on an upper bound of the sectional curvature, then M^n is homeomorphic to $\mathbb{S}^n$.

In this paper, the hypothesis on the Ricci curvature is replaced by the fact that M is isometrically immersed in a standard space form. Moreover, as we will see, the upper bound of the sectional curvature will be replaced by the L^∞-norm of the mean curvature or that of the second fundamental form. Recently, under the hypothesis that M is isometrically immersed in the Euclidean space, Colbois and Grosjean (see [3]) proved a pinching result on the first eigenvalue of the

Laplacian. More precisely, they proved that there exists a constant C depending on n and the L^∞-norm of the second fundamental form such that if $\frac{n}{V(M)^{1/p}}||H||_{2p}^2 - C < \lambda_1(M)$, then M is diffeomorphic to an n-dimensional sphere.

We keep on with studying hypersurfaces where little is known about pinching results. Indeed, we give pinching results for the extrinsic radius, which is the extrinsic analogue to the diameter, for hypersurfaces of the Euclidean space and hypersurfaces of the sphere and the hyperbolic space too.

For more convenience, we denote by $\mathcal{M}(n,\delta,R)$ *the set of all compact, connected and oriented n-dimensional Riemannian manifolds without boundary isometrically immersed into* $\mathbb{M}^{n+1}(\delta)$ *of extrinsic radius R and volume equal to* 1. In the case $\delta > 0$, we assume that M lies in an open hemisphere of $\mathbb{S}^{n+1}(\delta)$.

Theorem 1. *Let $(M^n,g) \in \mathcal{M}(n,\delta,R)$ and let p_0 be the center of the ball of radius R containing M. Then for any $\varepsilon > 0$, there exists a constant C_ε depending only on n, δ and the L^∞-norm of the mean curvature such that if*

$$t_\delta(R) < \frac{1}{||H||_\infty} + C_\varepsilon \qquad (P_{C_\varepsilon})$$

then

i) $\phi(M) \subset \overline{B}\big(p_0,R\big) \setminus B\big(p_0,R-\varepsilon\big)$.

ii) $\forall x \in S\big(p_0,R\big), \quad B(x,\varepsilon) \cap \phi(M) \neq \emptyset$.

Remark. We will see in the proof that $C_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow \infty$ or $\varepsilon \longrightarrow 0$.

We recall that the Haussdorff-distance between two compact subsets A and B of a metric space is given by

$$d_H(A,B) = \inf\big\{\eta \big|\ B \subset V_\eta(A) \quad \text{and} \quad A \subset V_\eta(B)\big\}$$

where for any subset A, $V_\eta(A)$ is the tubular neighborhood of A defined by $V_\eta(A) = \big\{x \big|\ \mathrm{dist}(x,A) < \eta\big\}$. So the points $i)$ and $ii)$ of Theorems 1 imply that

$$d_H\left(M, S(p_0,R)\right) \leq \varepsilon.$$

If the pinching condition is strong enough, with a control on the L^∞-norm of the second fundamental form instead of the L^∞-norm of the mean curvature, we obtain that M is diffeomorphic to a sphere and almost isometric to a geodesic sphere in the following sense:

Theorem 2. *Let $(M^n,g) \in \mathcal{M}(n,\delta,R)$ and let p_0 be the center of the ball of radius R containing M. Then there exists a constant C depending only on n, δ and the L^∞-norm of the second fundamental form such that if (P_C) is true, then M is diffeomorphic to $S(p_0,R)$.*

More preciesly, there exists a diffeomorphism F from M into the

geodesic hypersphere $S(p_0, R)$ of radius R which is a quasi-isometry. That is, for all $\theta \in]0,1[$, there exists a constant C depending on n, δ, $||B||_\infty$ and θ such that the pinching condition (P_C) implies

$$\Big| |dF_x(u)|^2 - 1 \Big| \leq \theta,$$

for all unit vector $u \in T_xM$.

Remark. In the two above Theorems, we assume that $V(M,g) = 1$. By homothety, we can deduce the same results for manifolds with arbitrary volume. Indeed, $(M, g') \in \mathcal{M}(n, \delta', R')$, with $g' = V(M)^{-2/n} g$, $\delta' = V(M)^{2/n}\delta$ and $R' = V(M)^{-1/n} R$.

We will see in Section 2 that in the case $\delta \geq 0$, Inequality (1) can be improved by replacing $||H||_\infty$ by $||H||_{2p}$ (see Proposition 2.2). Moreover, the equality is also caracterized by geodesic hyperspheres. Therefore, we can consider the corresponding pinching problem. Theorems 3 and 4 give the analogue of Theorems 1 and 2 for this integral lower bound.

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2. Preliminaries

First, let us introduce the following functions:

$$s_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}\, t) & \text{if } \delta > 0 \\ t & \text{if } \delta = 0 \\ \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta}\, t) & \text{if } \delta < 0 \end{cases}$$

and

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}\, t) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \\ \cosh(\sqrt{-\delta}\, t) & \text{if } \delta < 0 \end{cases}$$

We can easily check that $c_\delta^2 + \delta s_\delta^2 = 1$, $s_\delta' = c_\delta$ and $c_\delta' = -\delta s_\delta$. Moreover, we define the function $t_\delta = \dfrac{s_\delta}{c_\delta}$ which satisfies $t_\delta' = 1 + \delta t_\delta^2$.

Throughout this paper, we consider a Riemannian manifold (M^n, g) of $\mathcal{M}(n, \delta, R)$. The second fundamental form B of the immersion is defined by

$$B(X,Y) = \left\langle \overline{\nabla}_X \nu, Y \right\rangle,$$

where $< \cdot, \cdot >$ and $\overline{\nabla}$ are respectively the Riemannian metric and the Riemannian connection of $\mathbb{M}^{n+1}(\delta)$. The mean curvature of the immersion is

$$H = \frac{1}{n} \mathrm{tr}\,(B).$$

For any $p_0 \in \mathbb{M}^{n+1}(\delta)$ let $\exp_{p_0}$ be the exponential map at this point.

We consider $(x_i)_{1\leq i\leq n+1}$ the normal coordinates of $\mathbb{M}^{n+1}(\delta)$ centered at p_0. For $x\in\mathbb{M}^{n+1}(\delta)$, we denote by $r(x)=d(p_0,x)$ the geodesic distance from p_0 to x on $(\mathbb{M}^{n+1}(\delta),g_{can})$.

In what follows, ∇ and $\overline{\nabla}$ will be respectively the gradients associated to (M,g) and $(\mathbb{M}^{n+1}(\delta),g_{can})$. The corresponding Laplacians are Δ and $\overline{\Delta}$. The coordinates of $Z:=s_\delta(r)\overline{\nabla}r$ in the normal frame are $\left(\frac{s_\delta(r)}{r}x_i\right)_{1\leq i\leq n+1}$. We denote by X^T the projection of a vector field X on the tangent bundle of $\phi(M)$.

Now let's recall some properties of the exponential map. First, $\exp_{p_0}$ is a radial isometry, *i.e.*, for each $x\in\mathbb{M}^{n+1}(\delta)$, we have

$$\left\langle(\mathrm{d}\exp_{p_0})_X(X),(\mathrm{d}\exp_{p_0})_X(v)\right\rangle_x=\langle X,v\rangle_{p_0}\,,\tag{2}$$

where $X={\exp_{p_0}}^{-1}(x)$ and $v\in T_{p_0}\mathbb{M}^{n+1}(\delta)$. On the other hand, we have the following equalities (see Corollary 2.8 and Lemma 2.9 p153 in [18]). If v is a vector of $T_x\mathbb{M}^{n+1}(\delta)$ orthogonal to $\overline{\nabla}r$, we have

$$\left|\left(\mathrm{d}\exp_{p_0}{}^{-1}\right)_{|x}(v)\right|^2_{p_0}=\frac{r^2}{s^2_\delta(r)}|v|^2_x,\tag{3}$$

and

$$\left\langle\overline{\nabla}_v\overline{\nabla}r,v\right\rangle=\frac{c_\delta(r)}{s_\delta(r)}|v|^2.\tag{4}$$

Moreover, $\overline{\nabla}r$ is in the kernel of $\overline{\nabla}\mathrm{d}\,r$. In particular, for any $v\in T_x\mathbb{M}^{n+1}(\delta)$,

$$\left\langle\overline{\nabla}_{\overline{\nabla}r}\overline{\nabla}r,v\right\rangle=\left\langle\overline{\nabla}_v\overline{\nabla}r,\overline{\nabla}r\right\rangle=0.\tag{5}$$

Finally, we give the following lemma (see [11] or [9] for a proof):

Lemma 2.1. *i)* $\operatorname{div}(Z^T)=nc_\delta(r)+nH\langle Z,\nu\rangle\,,$

$$\textit{ii)}\ \delta\int_M g(Z^T,Z^T)dv_g\geq n\int_M\Big(c^2_\delta(r)-|H|c_\delta(r)s_\delta(r)\Big)dv_g,$$

Remark. Note that, after integration, the first point in the case $\delta=0$ is nothing else but the Hsiung-Minkowski formula (see [14]).

From this Lemma, we deduce the following estimates for the extrinsic radius in the case $\delta\geq 0$.

Proposition 2.2. *Let (M^n,g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into $\mathbb{R}^{n+1}$ or an open hemisphere of $\mathbb{S}^{n+1}(\delta)$. Then the extrinsic radius R of M satisfies*

$$t_\delta(R)\geq\frac{V(M)^{1/p}}{||H||_p},$$

for any $p\geq 1$. Moreover, equality holds if and only if (M^n,g) is a geodesic hypersphere of radius R.

Proof: After integration, the first point of Lemma 2.1 gives

$$\int_M c_\delta(r)dv_g \leq \int_M |H|s_\delta(r)dv_g.$$

Since s_δ is an increasing function and c_δ is a decreasing function, we get

$$c_\delta(R)V(M) \leq s_\delta(R)||H||_1.$$

The Hölder inequality gives the result for any L^p-norm. Obviously, if (M^n, g) is a geodesic hypersphere of radius R, then we have equality. Conversly, if equality holds, then

$$\int_M c_\delta(r)dv_g = c_\delta(R)V(M),$$

which implies that $r \equiv R$ and (M^n, g) is a geodesic hypersphere of radius R. $\square$

We have the following pinching results corresponding to this inequality.

Theorem 3. *Let $(M^n, g) \in \mathcal{M}(n, \delta, R)$ with $\delta \geq 0$ and let p_0 be the center of the ball of radius R containing M. Let $p \geq 1$. Then for any $\varepsilon > 0$, there exists a constant C_ε depending only on n, δ and the L^∞-norm of the mean curvature such that if*

$$(\widetilde{P}_{C_\varepsilon}) \qquad t_\delta(R) < \frac{1}{||H||_{2p}} + C_\varepsilon$$

then

i) $\phi(M) \subset \overline{B}\big(p_0, R\big) \setminus B\big(p_0, R-\varepsilon\big)$.

ii) $\forall x \in S\big(p_0, R\big), \quad B(x, \varepsilon) \cap \phi(M) \neq 0$.

Theorem 4. *Let $(M^n, g) \in \mathcal{M}(n, \delta, R)$ with $\delta \geq 0$ and let p_0 be the center of the ball of radius R containing M. Let $p \geq 1$. Then there exists a constant C depending only on n, δ and the L^∞-norm of the second fundamental form such that if*

$$(\widetilde{P}_C) \qquad t_\delta(R) < \frac{1}{||H||_{2p}} + C$$

then M is diffeomorphic and quasi-isometric to $S(p_0, R)$ in the sense of Theorem 2.

Remark. In the case $\delta \geq 0$, Theorems 1 and 2 are just corollaries of the two above theorems since if $V(M) = 1$, $||H||_\infty \geq ||H||_{2p}$.

3. An L^2-approach to pinching

A first step in the proof of the pinching results is to prove that the pinching condition (P_C) in the three cases, or $(\widetilde{P}_C)$, in the Euclidean or

spherical case implies that M is close to a hypersphere in an L^2-sense.

For this, let's consider the functions φ and ψ definied by

$$\varphi(r)=\begin{cases} t_\delta^2(R)-t_\delta^2(r) & \text{if} \quad \delta<0,\\ \\ s_\delta^2(R)-s_\delta^2(r) & \text{if} \quad \delta\geq 0\end{cases}$$

and

$$\psi(r)=c_\delta(r)|Z^T|.$$

3.1. **The Hyperbolic case.** In this section, we suppose $\delta<0$. Note that if the pinching constant C satisfies

$$C\leq\frac{1}{2}\left(\frac{1}{\sqrt{-\delta}}-\frac{1}{||H||_\infty}\right)=\alpha(||H||_\infty) \tag{6}$$

then (P_C) implies $t_\delta(R)\leq\frac{1}{2}\left(\frac{1}{\sqrt{-\delta}}+\frac{1}{||H||_\infty}\right)=\beta(||H||_\infty)<\frac{1}{\sqrt{-\delta}}$ and R is bounded from above by a constant depending only on $||H||_\infty$. In what follows, we assume that the pinching constant C satisfies the relation (6). We prove the following lemma

Lemma 3.1. *The pinching condition (P_C) with $C\leq\alpha(||H||_\infty)$ implies*

$$||\varphi||_2^2\leq A_1C,$$

where A_1 is a positive constant depending only on δ and $||H||_\infty$.

Proof: Since t_δ is an increasing function, we have

$$||\varphi||_2^2\leq t_\delta^2(R)\int_M\left(t_\delta^2(R)-t_\delta^2(r)\right).$$

Since $t_\delta^2(R)-t_\delta^2(r)\geq 0$ and $c_\delta(r)\geq 1$, so

$$\begin{aligned}\int_M\left(t_\delta^2(R)-t_\delta^2(r)\right) &\leq t_\delta^2(R)\int_M c_\delta^2(r)-\int_M s_\delta^2(r)\\ &\leq t_\delta^2(R)\int_M c_\delta^2(r)-\frac{1}{||H||_\infty^2}\int_M H^2 s_\delta^2(r).\end{aligned}$$

Using the Hölder inequality, we get

$$\int_M\left(t_\delta^2(R)-t_\delta^2(r)\right)\leq t_\delta^2(R)\int_M c_\delta^2(r)-\frac{1}{||H||_\infty^2}\frac{\left(\int_M|H|s_\delta(r)c_\delta(r)\right)^2}{\int_M c_\delta^2(r)}$$

Now using the relation $ii)$ of Lemma 2.1 and applying the pinching condition (P_C) with C satisfying (6), we find

$$\begin{aligned}\int_M\left(t_\delta^2(R)-t_\delta^2(r)\right) &\leq \left(t_\delta^2(R)-\frac{1}{||H||_\infty^2}\right)\int_M c_\delta^2(r)\\ &\leq \left(C^2+\frac{2C}{||H||_\infty}\right)c_\delta^2(R)\end{aligned}$$

and

$$||\varphi||_2^2 \;\leq\; s_\delta^2(R)\left(C^2+\frac{2C}{||H||_\infty}\right)\leq A_1 C,$$

where A_1 depends only on n, δ and $||H||_\infty$. $\square$

The next Lemma gives an upper bound for $||\psi||_2$ under the pinching condition.

Lemma 3.2. *The pinching condition (P_C) with $C \leq \alpha(||H||_\infty)$ implies*

$$||\psi||_2^2 \leq A_2 C + A_3||\varphi||_\infty,$$

where A_2 depends only on δ and A_3 depends on δ and $||H||_\infty$.

Proof: First, we observe that $|Z^T|^2 = |Z|^2 - \langle Z,\nu\rangle^2$. Since $|Z| = s_\delta(r)$, we have

$$||\psi||_2^2 \;\leq\; c_\delta^2(R)\left[s_\delta^2(R) - \frac{1}{||H||_\infty^2}\int_M \left(H^2\langle Z,\nu\rangle^2\right)\right].$$

Using Hölder inequality and Lemma 2.1 *i*), we get

$$\begin{aligned}
||\psi||_2^2 &\leq c_\delta^2(R)\left[s_\delta^2(R) - \frac{1}{||H||_\infty^2}\left(\int_M c_\delta(r)\right)^2\right].\\
&= c_\delta^2(R)\Bigg[s_\delta^2(R) - \frac{1}{||H||_\infty^2}c_\delta^2(R) - \frac{2c_\delta(R)}{||H||_\infty^2}\int_M (c_\delta(r)-c_\delta(R))\\
&\quad -\frac{1}{||H||_\infty^2}\left(\int_M c_\delta(r)-c_\delta(R)\right)^2\Bigg]\\
&\leq \left(\frac{2C}{||H||_\infty}+C^2\right)c_\delta^4(R) + K\big|c_\delta(r)-c_\delta(R)\big|
\end{aligned}$$

where K depends on δ and $||H||_\infty$. Since $t_\delta(R) \leq \beta(||H||_\infty) < \frac{1}{\sqrt{-\delta}}$, we deduce that there exists $K' > 0$ depending on n, δ and $||H||_\infty$ so that $\big|c_\delta(R) - c_\delta(r)\big| \leq K'||\varphi||_\infty$. This completes the proof. $\square$

3.2. **The Euclidean and Spherical cases.** Here, $\delta \geq 0$ and if $\delta > 0$ then we assume that $\phi(M)$ is contained in an open hemisphere (*i.e.* an open ball of radius $\frac{\pi}{2\sqrt{\delta}}$). First, note that the pinching condition $(\widetilde{P_C})$ with $C < 1$ and the fact that $V(M) = 1$ imply that there exist two constants α_n and β_n depending only on n so that $||H||_\infty \geq ||H||_{2p} \geq \alpha_n$ and $t_\delta(R) \leq \beta_n$. Consequently, R is bounded from above by a constant γ_n. That is an immediate consequence of the Sobolev following

inequality due to Hoffman and Spruck (cf [12] and [13]) for a nonnegative fonction f, by taking $f \equiv 1$

$$\left(\int_M f^{n/(n-1)} dv_g\right)^{2(n-1)/n} \leq K_n \left(\int_M |H|f + |\nabla f| dv_g\right).$$

Note that this inequality is true without further assumptions for $\delta = 0$. For $\delta > 0$, some conditions on the sectional curvature and the injectivity radius $i\big(\mathbb{S}^{n+1}(\delta)\big)$ of $\mathbb{S}^{n+1}(\delta)$ and on the support of the function f are needed. The first condition, $i\big(\mathbb{S}^{n+1}(\delta)\big) \geq \pi\delta^{-1}$, is satisfied since for the sphere $\mathbb{S}^{n+1}(\delta)$, we have $i\big(\mathbb{S}^{n+1}(\delta)\big) = \pi\delta^{-1}$. The second condition is $V(\mathrm{supp}(f)) \leq (1-\alpha)\omega_n\delta^{-n}$, for some $0 < \alpha < 1$ and where ω_n is the volume of the n-dimensional Euclidean ball. This condition is automatically satisfied if $\phi(M)$ lies in an open hemisphere.

In the sequel, we assume that the pinching constant satisfies $C < 1$. We have the following lemma

Lemma 3.3. *The pinching condition* $(\widetilde{P}_C)$ *with* $C < 1$ *implies*

$$||\varphi||_2^2 \leq \widetilde{A_1} C,$$

where $\widetilde{A_1}$ *is a positive constant depending only on* n *and* δ.

Proof: Since s_δ is an increasing function and c_δ is a decreasing function, we have

$$\begin{aligned} ||\varphi||_2^2 &\leq s_\delta^2(R) \int_M \left(s_\delta^2(R) - s_\delta^2(r)\right) \\ &\leq s_\delta^2(R) \left[t_\delta^2(R) \left(\int_M c_\delta(r)\right)^2 - \int_M s_\delta^2(r)\right] \end{aligned}$$

By the Hölder inequality, we have

$$||\varphi||_2^2 \leq s_\delta^2(R) \left[t_\delta^2(R) \left(\int_M c_\delta(r)\right)^2 - \frac{1}{||H||_{2p}^2} \left(\int_M H s_\delta(r)\right)^2\right]$$

Now using $i)$ of Lemma 2.1, we get

$$||\varphi||_2^2 \leq s_\delta^2(R) \left(\int_M c_\delta(r)\right)^2 \left[t_\delta^2(R) - \frac{1}{||H||_{2p}^2}\right] \leq \widetilde{A_1} C,$$

where $\widetilde{A_1}$ is a positive constant depending only on the dimension n (because $R \leq \gamma_n$) and δ. $\square$

The next lemma gives an upper bound for $||\psi||_2$ under the pinching condition.

Lemma 3.4. *The pinching condition* $(\widetilde{P}_C)$ *with* $C < 1$ *implies*

$$||\psi||_2^2 \leq \widetilde{A_2} C,$$

where $\widetilde{A_2}$ *is a positive constant depending only on* n *and* δ.

Proof: Since $|Z^T|^2 = |Z|^2 - \langle Z, \nu \rangle^2$, we have

$$\begin{aligned}
||\psi||_2^2 &\leq c_\delta^2(R) \left[\int_M s_\delta^2(R) - \frac{1}{||H||_{2p}^2} \left(\int_M H \langle Z, \nu \rangle \right)^2 \right] \\
&\leq c_\delta^2(R) \left[t_\delta^2(R) - \frac{1}{||H||_{2p}^2} \right] \left(\int_M c_\delta(r) \right)^2 \\
&\leq \widetilde{A_2} C.
\end{aligned}$$

Since $R \leq \gamma_n$, then $\widetilde{A_2}$ depends only on n and δ. $\square$

4. Proof of the Theorems

Let $(M^n, g) \in \mathcal{M}(n, \delta, R)$ and p_0 the center of the ball of radius R containing M. First, we need the following three lemmas

Lemma 4.1. *For any $\varepsilon > 0$, there exists C_ε depending on n, δ and $||H||_\infty$ so that if (P_{C_ε}) $\big($or $(\widetilde{P}_{C_\varepsilon})$ for $\delta \geq 0\big)$ is true, then*

$$\phi(M) \subset \overline{B}(p_0, R) \setminus B(p_0, R - \varepsilon).$$

Moreover, $C_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow +\infty$ or $\varepsilon \longrightarrow 0$.

We prove this lemma in Section 5. The second lemma is due to B. Colbois and J.F. Grosjean (see [3]).

Lemma 4.2. *Let x_0 be a point of the sphere $S(0, R)$ of $\mathbb{R}^{n+1}$. Assume that $x_0 = Ru$ where $u \in \mathbb{S}^n$. Now let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into $\mathbb{R}^{n+1}$ so that*

$$\phi(M) \subset \Big(B(p_0, R + \eta) \setminus B(p_0, R - \eta) \Big) \setminus B(x_0, \rho)$$

with $\rho = 4(2n - 1)\eta$ and suppose there exists a point $p \in M$ so that $\langle Z, u \rangle (p) \geq 0$. Then there exists $y_0 \in M$ so that the mean curvature $H(y_0)$ at y_0 satisfies $|H(y_0)| > \frac{1}{4n\eta}$.

Remark. Note that in [3], it is supposed that $\langle Z, u \rangle > 0$, but the condition $\langle Z, u \rangle \geq 0$ is sufficient.

We give a corresponding lemma for the hyperbolic and spherical cases.

Lemma 4.3. *Let x_0 be a point of the sphere $S(p_0, R)$ of $\mathbb{H}^{n+1}(\delta)$ (resp. an open hemisphere of $\mathbb{S}^{n+1}(\delta)$). Let $(M^n, g) \in \mathcal{M}(n, \delta, R)$ so that*

$$\phi(M) \subset \Big(\overline{B}(p_0, R) \setminus B(p_0, R - \eta) \Big) \setminus B(x_0, \rho)$$

with ρ such that

$$t_\delta\left((R + \rho)/2\right) - t_\delta(R/2) = 4(2n - 1)\eta$$

$$\Big(resp.\ t_\delta(R/2) - t_\delta\left((R-\rho)/2\right) = 4(2n-1)\eta \quad if\ \delta > 0\Big)$$

Then there exist two constants D and E depending on n, δ and R such that if $\eta \le D$, then there exists $y_0 \in M$ so that the mean curvature $H(y_0)$ satisfies

$$|H(y_0)| \ge \frac{E}{8n\eta}.$$

We prove this Lemma in Section 5. Now let us prove Theorems 1 and 3 using the above three Lemmas.

4.1. **Proof of Theorems 1 and 3.** For $\delta = 0$, the proof is an immediate consequence of Lemmas 4.1 and 4.2 and is similar to the proof of Theorem 1.2 in [3]. For $\delta \neq 0$, let $\varepsilon > 0$. We set $0 < \eta \le \inf\left\{D, \varepsilon, \frac{\gamma(\varepsilon)}{8(2n-1)}\right\}$, where

$$\gamma(\varepsilon) = \begin{cases} t_\delta\left(\frac{R+\varepsilon}{2}\right) - t_\delta\left(\frac{R}{2}\right) & \text{if } \delta < 0 \\ \\ t_\delta\left(\frac{R}{2}\right) - t_\delta\left(\frac{R-\varepsilon}{2}\right) & \text{if } \delta > 0. \end{cases}$$

Note that γ is an increasing smooth function with $\gamma(0) = 0$. From Lemma 4.1, there exists $K_\varepsilon = C_\eta$ such that (P_{K_ε}) implies

$$R - r \le \eta \le \varepsilon.$$

That's the first point of Theorems 1 and 3. Now let's assume that $\varepsilon < \gamma^{-1}\left(\frac{2E}{3||H||_\infty}\right)$. Suppose there exists $x \in S(p_0, R)$ such that $B(x,\varepsilon) \cap M = \emptyset$. Since $\gamma(\varepsilon) \ge 4(2n-1)\eta$, by Lemma 4.3, there exists a point $y_0 \in M$ so that

$$|H(y_0)| \ge \frac{E}{8n\eta} \ge \frac{(2n-1)E}{n\gamma(\varepsilon)} > ||H||_\infty.$$

Hence a contradiction and $B(x,\varepsilon) \cap M \neq \emptyset$. Moreover, by Lemma 4.1, $K_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow 0$ or $\varepsilon \longrightarrow 0$. $\square$

From Lemma 4.1, for any $\varepsilon > 0$, there exists C_ε depending on n, δ and $||H||_\infty$ so that if (P_{C_ε}) is true then $|R - r| \le \varepsilon$. Since $\alpha_n \le ||H||_\infty \le \frac{1}{\sqrt{n}}||B||_\infty$, we can assume that C_ε depends on n, δ and $||B||_\infty$.

For the proof of Theorems 2 and 4 we need the following lemma (which will be proved in Section 5) on the L^∞-norm of ψ.

Lemma 4.4. *For any $\varepsilon > 0$, there exists C_ε depending on n, δ and $||B||_\infty$ so that if (P_{C_ε}) $\big($or $(\widetilde{P}_{C_\varepsilon})$ for $\delta \ge 0\big)$ is true, then*

$$||\psi||_\infty \le \varepsilon.$$

Moreover, $C_\varepsilon \longrightarrow 0$ when $||B||_\infty \longrightarrow +\infty$ or $\varepsilon \longrightarrow 0$.

4.2. **Proof of Theorems 2 and 4.** Let $\varepsilon > 0$ such that $\varepsilon < t_\delta^{-1}\left(\frac{1}{||H||_\infty}\right) < R$. This choice of ε implies that if (P_{C_ε}) $\left(\text{or } (\widetilde{P}_{C_\varepsilon})\right)$ is true, then $r(x)$ never vanishes. So we can consider the following map

$$\begin{array}{rccl} F: & M & \longrightarrow & S(p_0, R(M)) \\ & x & \longmapsto & \exp_{p_0}\left(R\left(\mathrm{d}\exp_{p_0}\right)^{-1}(\overline{\nabla} r)\right). \end{array}$$

Let $X = {\exp_{p_0}}^{-1}(x)$. We can easily see that

$$\mathrm{d}\exp_{p_0}\big|_X(X) = |X|\overline{\nabla} r = r\overline{\nabla} r.$$

In the case of the Euclidean space ($\delta = 0$), $F(x)$ is precisely $R\frac{X}{|X|}$ where X is the position vector.

We will prove that F is a quasi-isometry. Indeed, we will prove that for any $\theta \in]0,1[$, there exists $\varepsilon(\theta)$ depending on n, δ, $||B||_\infty$ and θ such that for any $x \in M$ and any unit vector $u \in T_xM$, the pinching condition $(P_{C_{\varepsilon(\theta)}})$ implies

$$\left||\mathrm{d}_xF(u)|^2 - 1\right| \leq \theta.$$

For this, we compute $\mathrm{d}_xF(u)$ for a unit vector $u \in T_xM$. We have

$$\mathrm{d}F_x(u) = \mathrm{d}\exp_{p_0}\Big|_{R\frac{X}{|X|}}\left(R\,\mathrm{d}\left(\frac{X}{|X|}\right)_{\Big|_x}(u)\right) \tag{7}$$

Let $L(x) = \dfrac{X}{|X|} = \dfrac{{\exp_{p_0}}^{-1}(x)}{r}$. So we have

$$\mathrm{d}L_x(u) = \frac{1}{r}\mathrm{d}\exp_{p_0}\Big|_x^{-1}(u) - \frac{\mathrm{d}r(u)}{r^2}\exp_{p_0}^{-1}(x). \tag{8}$$

Using (7) and (8), we get

$$\mathrm{d}F_x(u) = \frac{R}{r}\mathrm{d}\exp\Big|_{R\frac{X}{|X|}}\left(\mathrm{d}\exp_{p_0}\Big|_x^{-1}(u)\right) - \frac{R}{r}\mathrm{d}r(u)\overline{\nabla} r\Big|_{F(x)} \tag{9}$$

We now compute $\left|\mathrm{d}_xF(u)\right|^2$. By (9) and the fact that $\exp_{p_0}$ is a radial isometry (see relation (2)), we have

$$\left|\mathrm{d}_xF(u)\right|^2 = \frac{R^2}{r^2}\left[\left|\mathrm{d}\exp_{p_0}\Big|_{R\frac{X}{|X|}}\left(\mathrm{d}\exp^{-1}_{p_0}\Big|_x(u)\right)\right|^2 - \mathrm{d}r(u)^2\right] \tag{10}$$

Let $v = u - \langle u, \nabla r\rangle\,\overline{\nabla} r$. That is, v is the part of u normal to $\overline{\nabla} r$. A straightforward calculation using (2) and (10) shows that

$$\left|\mathrm{d}_xF(u)\right|^2 = \frac{R^2}{r^2}\left|\mathrm{d}\exp_{p_0}\Big|_{R\frac{X}{|X|}}\left(\mathrm{d}\exp_{p_0}\Big|_x^{-1}(v)\right)\right|^2 \tag{11}$$

Finally, by (3), we have

$$\left|\mathrm{d}\exp_{p_0}{}^{-1}(v)\right|^2 = |v|^2\frac{r^2}{s_\delta^2(r)},$$

and by (3) again,

$$\left|\mathrm{d}_{\,x}F(u)\right|^2 = |v|^2\frac{s_\delta^2(R)}{s_\delta^2(r)}.$$

From now on, we consider the case $\delta < 0$, but the rest of the proof is similar to the case $\delta \geq 0$.
Since $|v|^2 = 1 - \langle u, \nabla r\rangle^2 \geq 1 - |\nabla r|^2$, we deduce that

$$\begin{aligned}\left|\left|\mathrm{d}_{\,x}F(u)\right|^2 - 1\right| &\leq \left|\frac{s_\delta^2(R)}{s_\delta^2(r)} - 1\right| + |\nabla r|\frac{s_\delta^2(R)}{s_\delta^2(r)}\\ &\leq s_\delta^2(r)\left|s_\delta^2(R) - s_\delta^2(r)\right| + \frac{s_\delta^2(R)}{c_\delta(r)s_\delta^3(r)}||\psi||_\infty.\end{aligned}$$

From Lemma 4.2, we know that for any $\eta > 0$, there exists a constant K_η so that (P_{K_η}) implies $||\psi||_\infty \leq \eta$. Moreover, since $C_\varepsilon \longrightarrow 0$ when $\varepsilon \longrightarrow 0$, there exists $\varepsilon \leq \eta$ depending on n, δ, $||H||_\infty$ and η so that $C_\varepsilon \leq K_\eta$, and then (P_{C_ε}) implies $||\psi||_\infty \leq \eta$. On the other hand, we have seen that R is bounded by a constant depending only on n, δ and $||H||_\infty$, then there exist three constants A_4, A_5 and A_6 depending on n, δ and $||H||_\infty$ so that

$$\begin{aligned}\left|\left|\mathrm{d}_{\,x}F(u)\right|^2 - 1\right| &\leq A_4||R - r||_\infty + A_5||\psi||_\infty\\ &\leq A_4\varepsilon + A_5\eta \leq A_6\eta\end{aligned}$$

Now, choosing $\eta = \frac{\theta}{A_6}$, we get

$$\left|\left|\mathrm{d}_{\,x}F(u)\right|^2 - 1\right| \leq \theta. \tag{12}$$

For $\theta \in]0,1[$, by (12), F is a local diffeomorphism from M to $S(p_0, R)$. Since for $n \geq 2$, $S(p_0, R)$ is simply connected, F is a diffeomorphism. Moreover, the relation (12) says that F is a quasi-isometry. □

5. Proof of the technical lemmas

The proof of Lemmas 4.1 and 4.4 is based on the following Proposition given by a Nirenberg-Moser's type argument.

Proposition 5.1. *Let (M^n, g) be a compact, connected, oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}(\delta)$ or an open hemisphere of $\mathbb{S}^{n+1}(\delta)$. Let ξ be a nonnegative continuous function on M such that ξ^k is smooth for $k \geq 2$. Let $0 \leq l < m \leq 2$ such that*

$$\frac{1}{2}\xi^{2k-2}\Delta\xi^2 \leq \operatorname{div}\omega + (\alpha_1 + k\alpha_2)\xi^{2k-l} + (\beta_1 + k\beta_2)\xi^{2k-m},$$

where ω is a 1-form and α_1, α_2, β_1, β_2 some nonnegative constants. Then for all $\eta > 0$, there exists a constant L depending only on α_1, α_2, β_1, β_2, $||H||_\infty$ and η such that if $||\xi||_\infty > \eta$ then

$$||\xi||_\infty \leq L||\xi||_2.$$

Moreover, L is bounded when $\eta \longrightarrow \infty$ and if $\beta_1 > 0$, $L \longrightarrow \infty$ when $||H||_\infty \longrightarrow \infty$ or $\eta \longrightarrow 0$.

This Proposition is proved in [3] in the Euclidean case. The proof in the hyperbolic and spherical cases is analogous, using the Sobolev inequality for hypersurfaces of $\mathbb{H}^{n+1}(\delta)$. Note that in the spherical case, M is assumed to be contained in an open ball of $\mathbb{S}^{n+1}(\delta)$ of radius less than $\frac{\pi}{2\sqrt{\delta}}$ as said in Section 3.2 (see [12] and [13] for details).

Proof of Lemma 4.1. We give the proof in the case $\delta \leq 0$. For the case $\delta \geq 0$ the same computations with φ give the result. First, we compute $\varphi^{2k-2}\Delta\varphi^2$.

$$\begin{aligned} \varphi^{2k-2}\Delta\varphi^2 &= 2\varphi^{2k-1}\Delta\varphi - 2|\nabla\varphi|^2\varphi^{2k-2} \\ &= -4t_\delta(r)(1+\delta t_\delta^2(r))\varphi^{2k-1}\Delta r - 2|\nabla\varphi|^2\varphi^{2k-2} \\ &\quad +4\varphi^{2k-1}\left|\nabla r\right|^2\Big(1+3\delta t_\delta^2(r)+2\delta^2 t_\delta^4(r)\Big) \end{aligned} \tag{13}$$

Let's compute

$$\begin{aligned} (1+\delta t_\delta^2)t_\delta\varphi^{2k-1}\Delta r &= -\mathrm{div}\left(\varphi^{2k-1}t_\delta(1+\delta t_\delta^2)\nabla r\right) \\ &\quad + \left\langle \nabla r, \nabla\left(\varphi^{2k-1}t_\delta(1+\delta t_\delta^2)\right)\right\rangle. \end{aligned} \tag{14}$$

Since $0 \leq t_\delta(r) \leq t_\delta(R) < \frac{1}{\sqrt{-\delta}}$ and $|\nabla r| \leq 1$, we deduce from the relations(13) and (14) that

$$\varphi^{2k-2}\Delta\varphi^2 \leq \mathrm{div}\,(\omega) + (\alpha_1 + k\alpha_2)\varphi^{2k-1} + (\beta_1 + k\beta_2)\varphi^{2k-2}, \tag{15}$$

where ω is a 1-form, α_1, α_2, β_1 and β_2 some nonnegative constants. We can apply Proposition 5.1 to the function φ with $l = 1$ and $m = 2$. We deduce that if $||\varphi||_\infty > \varepsilon$ then there exists a constant L such that

$$||\varphi||_\infty \leq L||\varphi||_2.$$

On the other hand, by Lemma 3.1, we know that if the pinching condition (P_C) is satisfied for $C \leq \alpha(||H||_\infty)$, then

$$||\varphi||_2^2 \leq A_1 C.$$

Take $C = C_\varepsilon = \inf\left\{\alpha(||H||_\infty), \frac{\varepsilon^2}{L^2 A_1}\right\}$. This choice implies

$$||\varphi||_\infty \leq \varepsilon,$$

that is, $t_\delta^2(R) - t_\delta^2(r) \leq \varepsilon$. Finally, we can choose C_ε smaller in order to have $R - r \leq \varepsilon$. $\square$

Proof of Lemma 4.4. We recall that $\psi = c_\delta(r)|Z^T|$ where $Z^T = s_\delta(r)\nabla r$. Note that

$$\nabla|Z|^2 = 2c_\delta(r)Z^T. \tag{16}$$

By using th Bochner formula, we the deduce that

$$\begin{aligned}
\frac{1}{2}\Delta\psi^2 &= \frac{1}{2}\Delta\Big|\nabla|Z|^2\Big|^2 \\
&= \Big\langle \nabla^*\nabla \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle - \Big|\nabla \mathrm{d}\,|Z|^2\Big|^2 \\
&= \Big\langle \Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle - \mathrm{Ric}\left(\mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\right) - \Big|\nabla \mathrm{d}\,|Z|^2\Big|^2 \\
&\leq \Big\langle \Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle - \mathrm{Ric}\left(\mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\right)
\end{aligned}$$

Now, with the Gauss formula, we can express the Ricci curvature in terms of the second fondamental form B. Precisely, we have

$$\begin{aligned}
\frac{1}{2}\Delta\psi^2 &\leq \Big\langle \Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle - nH\left\langle B\nabla|Z|^2, \nabla|Z|^2\right\rangle \\
&\quad + \Big|B\nabla|Z|^2\Big|^2 - 4\delta\Big|\nabla|Z|^2\Big|^2 \\
&\leq \Big\langle \Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle - 4nHc_\delta^2(r)\left\langle BZ^T, Z^T\right\rangle \\
&\quad + 4c_\delta^2(r)|BZ^T|^2 - 4(n-1)c_\delta^2(r)\delta|Z^T|^2
\end{aligned}$$

Since $|Z^T| \leq s_\delta(R)$, we easily see that the pinching condition (P_C), with $C < 1$ for $\delta \geq 0$ or $C < \alpha(||H||_\infty)$ for $\delta < 0$, implies

$$||\psi||_\infty \leq K_1,$$

where K_1 is a positive constant depending only on n, δ and $||B||_\infty$. It follows that

$$\frac{1}{2}\left(\Delta\psi^2\right)\psi^{2k-2} \leq \Big\langle \Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle \psi^{2k-2} + K_2\psi^{2k-2}.$$

Let $\omega = \Delta|Z|^2\psi^{2k-2}\mathrm{d}\,|Z|^2$. We have

$$\begin{aligned}
\Big\langle \Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\Big\rangle \psi^{2k-2} &= \mathrm{div}\,(\omega) + (\Delta|Z|^2)^2\psi^{2k-1} \\
&\quad -2(2k-2)c_\delta(r)\Delta|Z|^2\left\langle Z^T, \mathrm{d}\,\psi\right\rangle \psi^{2k-3},
\end{aligned}$$

Moreover, a straightforward calculation using the facts that

$$e_i(|Z^T|) = \frac{1}{2}\frac{e_i(|Z^T|^2)}{|Z^T|},$$

and $|Z^T|^2 = s_\delta^2(r) - \langle Z, \nu\rangle^2$ gives

$$\begin{aligned}
e_i(\psi) &= c_\delta(r)\frac{\delta c_\delta(r)s_\delta(r)e_i(r) - \langle Z,\nu\rangle\left(B_{ij}\langle Z, e_j\rangle + \left\langle \overline{\nabla}_{e_i}Z, \nu\right\rangle\right)}{|Z^T|} \\
&\quad -\delta s_\delta(r)|Z^T|e_i(r)
\end{aligned}$$

All the terms can be bounded easily except $\left\langle\overline{\nabla}_{e_i} Z, \nu\right\rangle$ which will be investigated. Since $Z = s_\delta(r)\overline{\nabla}r$, this is equivalent to have an upper bound for $\left\langle\overline{\nabla}_{e_i}\overline{\nabla}r, \nu\right\rangle$. From (4) and (5), we deduce that

$$\left|\left\langle\overline{\nabla}_{e_i}\overline{\nabla}r, \nu\right\rangle\right| = \frac{c_\delta(r)}{s_\delta(r)}\left\langle e_i^t, \nu^t\right\rangle,$$

where e_i^t and ν^t are the part of e_i and ν tangent to the geodesic sphere of radius r, that is, orthogonal to $\overline{\nabla}r$. Since

$$|\nu^t|^2 = 1 - \left\langle\nu, \overline{\nabla}r\right\rangle^2 = |\nabla r|^2,$$

we have

$$\left|\left\langle\overline{\nabla}_{e_i}\overline{\nabla}r, \nu\right\rangle\right| \leq \frac{c_\delta(r)}{s_\delta(r)}|\nabla r|^{\cdot}$$

Then

$$\begin{aligned}
\left\langle\Delta \mathrm{d}\,|Z|^2, \mathrm{d}\,|Z|^2\right\rangle\psi^{2k-2} \leq\ & \operatorname{div}(\omega) + (\Delta|Z|^2)^2\psi^{2k-1} \\
& + 2(2k-2)\Big|\Delta|Z|^2\Big|\psi^{2k-2} \\
& + K_3\Big|\Delta|Z|^2\Big|\,||B||_\infty\psi^{2k-2},
\end{aligned}$$

where, K_3 depends on n, δ and $||H||_\infty$. Moreover, $\Delta|Z|^2 = -2\operatorname{div}\left(c_\delta(r)Z^T\right)$ and by Lemma 2.1 *i*), we deduce that there exists a constant K_4 depending on n, δ and $||H||_\infty$ such that

$$\Delta|Z|^2 \leq K_4.$$

Finally, we have

$$\psi^{2k-2}\Delta\psi^2 \leq \operatorname{div}(\omega) + (\alpha_3 + k\alpha_4)\psi^{2k-1} + (\beta_3 + k\beta_4)\psi^{2k-2}$$

with some nonnegative constants α_3,α_4,β_3 and β_4 depending on n, δ and $||B||_\infty$. Now applying Proposition 5.1 with $l = 1$ and $m = 2$, we get that for $\eta > 0$, there exists L depending on n, δ, $||B||_\infty$ and η so that if $||\psi||_\infty > \eta$ then

$$||\psi||_\infty \leq L||\psi||_2.$$

From Lemma 3.2 we deduce that if (P_C) holds with $C < 1$ for $\delta \geq 0$ or $C < \alpha(||H||_\infty)$ for $\delta < 0$, then

$$||\psi||_2^2 \leq A_2 C + A_3||\varphi||_\infty.$$

Let $\varepsilon > 0$, and put $K_\varepsilon := \inf\left\{\frac{\varepsilon^2}{2L^2A_2}, C_{\frac{\varepsilon^2}{2L^2A_3}}\right\}$ where C is the constant defined in the proof of the Lemma 4.1. Then, if (P_{K_ε}) holds, we have

$$||\psi||_\infty \leq L||\psi||_2 \leq \varepsilon.$$

$\square$

Proof of Lemma 4.3. We give the proof in the hyperbolic case, $\delta = -1$. The proof for any $\delta < 0$ or for the spherical case is similar.

Let us consider f the conformal map from the unit ball $\widetilde{B}(0,1)$ of $\mathbb{R}^{n+1}$ into $\mathbb{H}^{n+1}$ so that $f(0) = p_0$. The conformal factor is the function h^2 where h is defined by

$$\begin{array}{rccl} h\,: & [0,1[& \longrightarrow & \mathbb{R}_+ \\ & r & \longmapsto & \frac{2}{1-r^2} \end{array}$$

For any $\rho > 0$, $B(p_0,\rho) = f\left(\widetilde{B}(0,\widetilde{\rho})\right)$, where $\widetilde{B}(0,\widetilde{\rho}) \subset \widetilde{B}(0,1)$ is the ball of radius $a(\rho) := \widetilde{\rho} = t_\delta(\rho/2)$. Let $\widetilde{\phi} = f^{-1}\circ\phi$. By hypothesis,

$$\phi(M) \subset \Big(B(p_0, R+\eta) \setminus B(p_0, R-\eta)\Big) \setminus B(x_0,\rho),$$

with $x_0 \in S(p_0,R)$ and ρ chosen so that

$$a(R+\rho) - a(R) = 4(2n-1)\eta,$$

then

$$\widetilde{\phi}(M) \subset \Big(B\big(p_0, a(R+\eta)\big) \setminus B\big(p_0, a(R-\eta 0\big)\Big) \setminus B(z_0,\rho'),$$

where $z_0 = \frac{1}{2}\left[a(R+\rho) + a(R-\rho)\right]u$, with u a unit vector and

$$\rho' = \frac{1}{2}\left[a(R+\rho) - a(R-\rho)\right].$$

Obviously we have

$$\widetilde{B}\big(0, a(R+\eta)\big) \setminus \widetilde{B}\big(0, a(R-\eta)\big) \subset \widetilde{B}\big(0, a(R)+\eta\big) \setminus \widetilde{B}\big(0, a(R)-\eta\big).$$

Moreover, by concavity of the function a, we have

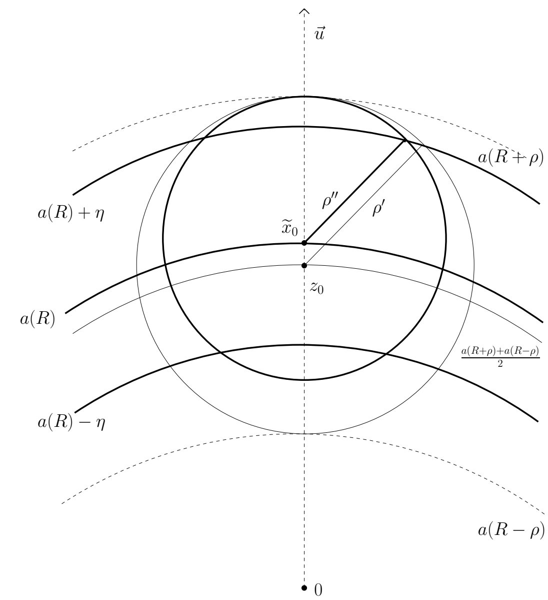
$$a(R+\rho) + a(R-\rho) \leq 2a(R),$$

and then

$$\widetilde{B}(z_0,\rho') \supset \widetilde{B}(\widetilde{x_0},\rho''),$$

where $\widetilde{x_0} = a(R)u$ and $\rho'' = a(R+\rho) - a(R)$. Finally, we have

$$\widetilde{\phi}(M) \subset \Big[\widetilde{B}\big(0, a(R)+\eta\big) \setminus \widetilde{B}\big(0, a(R)-\eta\big)\Big] \setminus \widetilde{B}\big(\widetilde{x_0}, 4(2n-1)\eta\big).$$



Since $a(R)$ is the extrinsic radius of $\widetilde{\phi}(M)$, there exists a point $p \in \widetilde{\phi}(M)$ so that $\left\langle \widetilde{Z}, u \right\rangle (p) \geq 0$, where $\widetilde{Z}$ is the position vector of $\widetilde{\phi}(M)$ in $\widetilde{B}(0,1)$. By Lemma 4.2, there exists $y_0 \in \widetilde{\phi}(M)$ so that the mean curvature $\widetilde{H}$ satisfies $|\widetilde{H}(y_0)| > \frac{1}{4n\eta}$. Moreover, we have the well-known formula for the conformal mean curvature (see for example [8])

$$H = h^{-1}\left(\widetilde{H} + h^{-1}\left\langle \widetilde{\nabla} h, \widetilde{\nu} \right\rangle\right),$$

where $\widetilde{\nabla}$ and $\widetilde{\nu}$ are the gradient and the normal unit vector field in $\widetilde{B}(0,1)$, and $\langle .,. \rangle$ is the Euclidean scalar product in $\widetilde{B}(0,1)$. Therefore,

$$|H| \geq h^{-1}(\tilde{r})\left(\widetilde{H} - h^{-1}(\tilde{r})|\tilde{\nabla} h|\right),$$

where $\widetilde{r}(x)$ is the Euclidean distance from 0 to x. So we have

$$\frac{1}{2} \geq h^{-1}(\tilde{r}) = \frac{1-\tilde{r}^2}{2} \geq \frac{1-a(R)^2}{2}, \tag{17}$$

and

$$|\widetilde{\nabla} h| = \frac{4}{\left(1-\tilde{r}^2\right)^2}|\tilde{r}\widetilde{\nabla}\tilde{r}| \leq \frac{4}{\left(1-a(R)^2\right)^2}. \tag{18}$$

Finally, by (17) and (18), we get

$$|H(f^{-1}(y_0))| \geq \frac{E}{4n\eta} - \frac{1}{4E^2},$$

where $E = \frac{1-a(R)^2}{2}$ is a constant depending on n, δ and R. Moreover, there exists a constant D depending on n, δ and R so that if $\eta \leq D$, then $\frac{1}{4E^2} \leq \frac{E}{8n\eta}$, and so

$$|H(f(y_0))| \geq \frac{E}{8n\eta}.$$

□

Remark. If we suppose (P_C) with $C < \alpha(||H||_\infty)$ for $\delta < 0$ (*resp.* with $C < 1$ for $\delta \geq 0$) then D and E depend on n, δ and $||H||_\infty$ (*resp.* on n and δ).

Remark. For $\delta > 0$, the function a is convex and so $\rho'' = a(R) - a(R-\rho)$.

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